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Complementary fields conservation equation derived from the scalar wave equation

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Abstract

A conservation equation for the scalar wave equation is derived from two linearly independent solutions. In the one-dimensional limit the conservation equation yields a previously known invariant. The continuity equation derived for a complex disturbance is shown to yield an equivalent result. The obtention of the second independent solution is discussed using two different schemes that lead either to orthogonal trajectories or to derivative fields. The complementary fields may be visualized as out-of-phase fields where a negative-valued density is interpreted in terms of the leading or lagging field. These results are compared with the usual definition of energy density and flow for scalar waves. In the monochromatic plane wave case, the averages of all the proposed densities and flows converge to the same result. The physical meaning of the different approaches is discussed.

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1. Introduction

The existence of conserved quantities is of fundamental importance in almost every field of physics. Under appropriate circumstances, conserved quantities become invariants of motion. A wide variety of mathematical methods have been developed in order to obtain invariants that span from elementary algebraic methods to symmetry considerations evaluated through symplectic group transformations or Noether's theorem [1]. An orthogonal functions invariant, closely related to the Ermakov–Lewis invariant, has been recently derived for the classical time-dependent harmonic oscillator (TDHO) equation [2]. A quantum version of this constant of motion may be used, for example, to solve the one-dimensional TDHO Schrödinger equation [3].

In the present paper, the orthogonal functions derivation is extended to the (3 + 1)-dimensional scalar wave equation. For systems with one degree of freedom, the assessed quantity becomes the previously known invariant as shown in section 2. In the case of a complex perturbation, exposed in section 3, the conserved quantity and its flow are shown to be equivalent to their counterparts in the previous real orthogonal functions derivation. These expressions have been studied before in the context of the Klein–Gordon equation [4]. Due to the lack of a positive definite density and thus the impossibility of representing a probability density, this continuity equation has not received much attention. Nonetheless, this charge-like density will be shown to emanate in the presence of two out-of-phase fields. This interpretation is plausible for wave phenomena where the disturbance comes from the imbalance between two forms of energy [5]. The complementary field or linearly independent function is then evaluated from a given solution using two different procedures. The first one, described in section 4.1, proposes a generalization of the linearly independent solution obtained in the one-dimensional case. The second procedure, described in section 4.2, evaluates the temporal derivative of the field in order to produce a second solution. In section 5, the density and flow obtained from these results are compared with the positive definite density conservation equation usually invoked to evaluate the energy content of a wave fulfilling a scalar second-order differential equation. A summary of results together with the conclusions is presented in the last section.

2. Complementary functions procedure

In order to obtain a continuity equation of the form $\nabla \cdot \mathbf{J} + (\partial/\partial t)\rho = 0$, consider the following procedure. The starting point is the scalar wave equation

$$\nabla^2 \psi(\mathbf{r}, t) - \frac{1}{v^2}\frac{\partial^2 \psi(\mathbf{r}, t)}{\partial t^2} = 0 \tag{1}$$

where the scalar ψ represents the disturbance and v is the velocity of propagation. Allow for two real linearly independent solutions of the wave equation to be $\psi_1(\mathbf{r}, t)$ and $\psi_2(\mathbf{r}, t)$ (the notation that exhibits the space and time dependence $(\mathbf{r}, t)$ is dropped in the derivation and shown explicitly only when needed). Perform the product of ψ_2 and the wave equation for ψ_1:

$$\psi_2 \left(\nabla^2 \psi_1 - \frac{1}{v^2}\frac{\partial^2 \psi_1}{\partial t^2} \right) = 0.$$

Calculate the product inverting the solutions and evaluate their difference:

$$(\psi_2 \nabla^2 \psi_1 - \psi_1 \nabla^2 \psi_2) + \frac{1}{v^2}\left(\psi_1 \frac{\partial^2 \psi_2}{\partial t^2} - \psi_2 \frac{\partial^2 \psi_1}{\partial t^2} \right) = 0.$$

This equation, with the aid of Green's theorem, may be written as

$$\nabla \cdot (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2) + \frac{1}{v^2}\frac{\partial}{\partial t}\left(\psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t} \right) = 0 \tag{2}$$

where the assessed quantity ψ_ρ and its corresponding flux $\triangleright \psi_\rho$ are defined as

$$\psi_\rho \equiv \left(\psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t} \right) \qquad \triangleright \psi_\rho \equiv (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2). \tag{3}$$

Provided that the medium does not exhibit dispersion but is allowed to be inhomogeneous, the above expression may be written as a continuity equation

$$\nabla \cdot \mathbf{S} + \frac{\partial}{\partial t}\mathcal{E} = 0 \tag{4}$$

where the density of the conserved quantity and its associated flux are

$$\mathcal{E} = \frac{1}{v^2(\mathbf{r})}\left(\psi_1\frac{\partial\psi_2}{\partial t} - \psi_2\frac{\partial\psi_1}{\partial t}\right) \qquad \mathbf{S} = (\psi_2\nabla\psi_1 - \psi_1\nabla\psi_2). \tag{5}$$

These definitions may be scaled by an arbitrary constant; in particular, a factor of $1/2$ will be included when these quantities are compared with the usual energy and momentum flow variables. If the medium were homogeneous but with a time-dependent velocity, a continuity equation may also be obtained,

$$\nabla \cdot [v^2(t)(\rhd\, \psi_\rho)] + \frac{\partial}{\partial t}(\psi_\rho) = 0 \tag{6}$$

provided that the density and flux are now defined by $[\mathcal{E}]_{\text{disp}} = \psi_\rho$ and $[\mathbf{S}]_{\text{disp}} = v^2(t)\ (\rhd\, \psi_\rho)$, respectively. In a homogeneous dispersionless medium either definition may, of course, be used. The quantities in (3) may be written as

$$\psi_\rho = \psi_1^2\frac{\partial}{\partial t}\left(\frac{\psi_2}{\psi_1}\right) = -\psi_2^2\frac{\partial}{\partial t}\left(\frac{\psi_1}{\psi_2}\right) \qquad \rhd\, \psi_\rho = -\psi_1^2\nabla\left(\frac{\psi_2}{\psi_1}\right)$$

so that any one solution may be readily expressed in terms of the other function and the density, i.e. $\psi_2 = \psi_1 \int \left(\psi_\rho/\psi_1^2\right) \mathrm{d}t$. Linear independence of the solutions in the temporal variable is ensured if the scalar field density does not vanish, $\psi_\rho \neq 0$. Whereas linear independence in the spatial variables is obtained if the vector field $\rhd\, \psi_\rho \neq 0$. The wave equation ensures that if the two fields are linearly independent in the temporal variable then they are also independent in the spatial variables. The only trivial exception being if the velocity of propagation is zero. Hereafter, we shall refer to these linearly independent solutions in the temporal and spatial variables as *complementary fields*.

Orthogonality, in the analytic functions sense, of the linearly independent solutions $\psi_1(\mathbf{r}, t)$ and $\psi_2(\mathbf{r}, t)$ over the interval $[a, b]$ with respect to the weight function w may be achieved through Schmidt's method. Given the function $\psi_2(\mathbf{r}, t)$, the orthogonal solution in the time variable $\psi_{1\perp}(\mathbf{r}, t)$ is given by $\psi_{1\perp} = \psi_1 - \lambda_{12}\psi_2$, where

$$\lambda_{12} = \frac{\int_a^b \psi_2\psi_1 w\, \mathrm{d}t}{\int_a^b \psi_2\psi_2 w\, \mathrm{d}t}.$$

The fields $\psi_{1\perp}(\mathbf{r}, t)$ and $\psi_2(\mathbf{r}, t)$ then obey the relationship $\int_a^b \psi_{1\perp}\psi_2 w\, \mathrm{d}t = 0$. Biorthonormal systems of this sort have been used to identify the radiative and nonradiative parts of a wave field [6]. If the density is evaluated using the linearly independent solution ψ_1 albeit not necessarily orthogonal to ψ_2, then

$$\psi_\rho = -\psi_2^2\frac{\partial}{\partial t}\left(\frac{\psi_1}{\psi_2}\right) = -\psi_2^2\frac{\partial}{\partial t}\left(\frac{\psi_{1\perp} + \lambda_{12}\psi_2}{\psi_2}\right) = -\psi_2^2\frac{\partial}{\partial t}\left(\frac{\psi_{1\perp}}{\psi_2}\right).$$

Therefore, the contribution of the non-orthogonal component is zero and thus *the non-vanishing contribution to the density comes from the orthogonal fields solutions*. An analogous procedure may be performed in the spatial domain in order to derive the orthogonal spatial field. The non-zero contribution to the flow is again obtained from the spatially orthogonal field solution. The complementary field function is, of course, not unique since any linearly dependent function may be added without altering the density function ψ_ρ. In addition, as is well known, a continuity equation admits a density that is defined up to a time-independent scalar $\mathcal{E}' = \mathcal{E} + G(\mathbf{r})$ and a flow with an arbitrary divergence-free field $\mathbf{S}' = \mathbf{S} + \nabla \times \mathbf{G}$. Furthermore, for any twice differentiable vector field $\mathbf{G}$, a modified density $\mathcal{E}' = \mathcal{E} + \nabla \cdot \mathbf{G}$ and flux $\mathbf{S}' = \mathbf{S} - \partial\mathbf{G}/\partial t$ are also admissible [7].

2.1. Systems with restricted degrees of freedom

2.1.1. Spatially harmonic field. In the particular case where the field functions are harmonic in the spatial domain

$$\nabla^2\psi(\mathbf{r},t) = -k^2(t)\psi(\mathbf{r},t) \tag{7}$$

where the wave vector magnitude $k^2(t)$ is spatially constant but has an arbitrary time dependence. The complementary functions procedure applied to (7) yields $\nabla\cdot(\triangleright\ \psi_\rho) = 0$. The continuity equation (2) then leads to an invariant

$$\psi_\rho \to Q = \psi_1\frac{\partial\psi_2}{\partial t} - \psi_2\frac{\partial\psi_1}{\partial t}. \tag{8}$$

Substitution of the spatial harmonic dependence (7) in the wave equation (1) gives

$$k^2(t)\psi(\mathbf{r},t) + \frac{1}{v^2(t)}\frac{\partial^2\psi(\mathbf{r},t)}{\partial t^2} = 0. \tag{9}$$

The standard separation of variables $\psi(\mathbf{r},t) = \psi_{\mathrm{sp}}(\mathbf{r})\psi_t(t)$ then yields decoupled differential equations for the spatial and temporal behaviour. The temporal equation is equal to the time-dependent oscillator equation, with time-dependent parameter given by $\Omega^2(t) = k^2(t)v^2(t)$. Therefore, the harmonic spatial field restriction, as is well known, describes a continuum problem that fulfils a temporal differential equation identical to that obtained from the discrete non-propagating problem of a single particle in a time-dependent potential. The density obtained in the different formalisms under this restriction will be discussed in the following sections.

Given one field solution, say $\psi_{t1}(t)$, the complementary field may be readily obtained from the invariant relationship (8) in this one-dimensional case,

$$\psi_{t2}(t) = \psi_{t1}(t)\int\frac{Q}{\psi_{t1}^2(t)}\,\mathrm{d}t.$$

This is the common way of obtaining, with the aid of the Wronskian Q, a linearly independent solution in ordinary second-order differential equations [8]. In terms of real amplitude 'a' and phase 's' variables, the one-dimensional exact invariant is $Q = a^2(t)\,\mathrm{d}s(t)/\mathrm{d}t$. The Ermakov–Lewis exact invariant may be derived from the general version of this constant of motion [2]. Given a solution of the form $\psi_{1t}(t) = a(t)\cos[s(t)]$, the linearly independent solution is

$$\psi_{t2}(t) = a(t)\cos[s(t)]\int\frac{a^2(t)\,\mathrm{d}s(t)/\mathrm{d}t}{a^2(t)\cos^2[s(t)]}\,\mathrm{d}t = a(t)\sin[s(t)]. \tag{10}$$

2.1.2. Monochromatic field. A field with harmonic time dependence

$$\frac{\partial^2\psi}{\partial t^2} = -\omega_0^2\psi \tag{11}$$

transforms the wave equation into the time-independent diffusion (or Helmholtz) equation

$$\nabla^2\psi(\mathbf{r}) = -\frac{\omega_0^2}{v^2(\mathbf{r})}\psi(\mathbf{r})$$

where the medium inhomogeneity $\kappa^2(\mathbf{r}) = \omega_0^2/v^2(\mathbf{r})$ is now constant in time but has an arbitrary spatial dependence. The solution in amplitude and phase variables is given by

$$\psi_1(\mathbf{r},t) = a(\mathbf{r})\cos(\phi(\mathbf{r}) - \omega_0 t + \varphi_0) \tag{12}$$

where φ_0 is a constant phase. Since the continuity equation then reads $\nabla \cdot (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2) = 0$, the obtention of the linearly independent solution is not straightforward from this result. Nonetheless, the harmonic time dependence yields a time-independent density and thus (8) is fulfilled. The linearly independent solution may then be proposed to be

$$\psi_2(\mathbf{r}, t) = a(\mathbf{r}) \sin(\phi(\mathbf{r}) - \omega_0 t). \tag{13}$$

The density and flow are then

$$\psi_\rho = -a^2(\mathbf{r})\omega_0 \qquad \triangleright\ \psi_\rho = -a^2(\mathbf{r})\nabla\phi(\mathbf{r}). \tag{14}$$

According to this expression, the flow is finite in any direction where the wave vector has a non-vanishing projection. Note that the general solution (12) and the linearly independent solution (13) are being used to obtain the density and its corresponding flow. The usual sequence in ordinary differential equations is the opposite where the general solution is obtained from a particular solution with the help of a constant density. In an unrestricted system, the density becomes in general spatially and time dependent. The problem then will be to find an independent solution in order to evaluate the density as we shall see in section 4. The continuity equation for a monochromatic field in amplitude and phase variables is from (4) and (14),

$$\nabla \cdot \mathbf{S} = \nabla \cdot [a^2(\mathbf{r})\nabla\phi(\mathbf{r})] = 0 \tag{15}$$

since the density is time independent. The paraxial approximation of this expression has been used to retrieve the phase from intensity measurements in optical testing [9] and other inverse source problems [10]. The one-dimensional restriction, say in the z direction, again reduces to an invariant of the form $Q = a^2(z)(\partial/\partial z)\phi(z)$. This invariant allows for the decoupling of the amplitude and phase equations leading to an Ermakov-type equation. The differential equation is once more formally equivalent to the time-dependent harmonic oscillator equation with the time variable replaced by the spatial coordinate. This equation describes, for example, the propagation of an electromagnetic wave at normal incidence in an arbitrary inhomogeneous medium [11].

3. Complex disturbance

Permit the disturbance to be complex $\psi \rightarrow \tilde{\psi}$. The most general solution has the form

$$\tilde{\psi}_g = (b_{1r}\psi_1 + b_{2r}\psi_2) + (b_{1i}\psi_1 + b_{2i}\psi_2)\mathrm{i}$$

where the coefficients $b_{1r}, b_{2r}, b_{1i}, b_{2i}$ are constant. The complementary field is obtained from the substitutions $\psi_1 \rightarrow \psi_2$ and $\psi_2 \rightarrow \psi_1$, where the constant coefficients may, in general, be different from the previous ones,

$$\tilde{\psi}_g^{(\mathrm{comp})} = (b_{c1r}\psi_1 + b_{c2r}\psi_2) + (b_{c1i}\psi_1 + b_{c2i}\psi_2)\mathrm{i}.$$

It may be worked out that the density $\tilde{\psi}_g(\partial/\partial t)\tilde{\psi}_g^{(\mathrm{comp})} - \tilde{\psi}_g^{(\mathrm{comp})}(\partial/\partial t)\tilde{\psi}_g$ is proportional to $\psi_1(\partial/\partial t)\psi_2 - \psi_2(\partial/\partial t)\psi_1$, the proportionality constant being

$$(b_{c2i} - \mathrm{i}b_{c2r})(b_{1i} - \mathrm{i}b_{1r}) - (b_{c1i} - \mathrm{i}b_{c1r})(b_{2i} - \mathrm{i}b_{2r}).$$

Equivalent results are obtained in the spatial domain for the flow. It may be seen from this result that in the complex case, just as in the real disturbance derivation, there is no contribution to the density or flow for complementary field terms that are linearly dependent on the reference field. For example, if only b_{1r} is different from zero in the $\tilde{\psi}_g$ solution, the only non-vanishing contribution comes from the terms involving ψ_2 in the complementary solution $\tilde{\psi}_g^{(\mathrm{comp})}$, i.e. the coefficients b_{c2i}, b_{c2r}. It is therefore sufficient, without loss of generality, to introduce linearly independent complex solutions; let the choice be $b_{1r} = b_{c1r}$ and $b_{2i} = -b_{c2i}$ while all other constants are set to zero. The complex complementary fields are then

$$\tilde{\psi} = b_{1r}\psi_1 + b_{2i}\psi_2\mathrm{i} \qquad \tilde{\psi}^{(\mathrm{comp})} = b_{1r}\psi_1 - b_{2i}\psi_2\mathrm{i}. \tag{16}$$

These two solutions are linearly independent provided that b_{1r} and b_{2i} are non-zero; that is, the solutions must neither be purely real nor purely imaginary. To wit, the density

$$\tilde{\psi}_\rho = 2\mathrm{i}b_{1r}b_{2i}\left(\psi_1\frac{\partial\psi_2}{\partial t} - \psi_2\frac{\partial\psi_1}{\partial t}\right)$$

is finite since ψ_1, ψ_2 are linearly independent solutions. There is an alternative procedure that may be employed to obtain this result if the complementary functions procedure used to derive a continuity equation is recreated using the pair $\tilde{\psi}^*$ and $\tilde{\psi}$ rather than the linearly independent real solutions ψ_1, ψ_2. The continuity equation then reads

$$\nabla\cdot(\tilde{\psi}\nabla\tilde{\psi}^* - \tilde{\psi}^*\nabla\tilde{\psi}) + \frac{1}{v^2}\frac{\partial}{\partial t}\left(\tilde{\psi}^*\frac{\partial\tilde{\psi}}{\partial t} - \tilde{\psi}\frac{\partial\tilde{\psi}^*}{\partial t}\right) = 0 \tag{17}$$

where the assessed quantity ψ_ρ and its corresponding flux $\triangleright\ \psi_\rho$ are now defined by

$$\psi_\rho = \frac{1}{2\mathrm{i}}\left(\tilde{\psi}^*\frac{\partial\tilde{\psi}}{\partial t} - \tilde{\psi}\frac{\partial\tilde{\psi}^*}{\partial t}\right) \qquad \triangleright\ \psi_\rho = \frac{1}{2\mathrm{i}}(\tilde{\psi}\nabla\tilde{\psi}^* - \tilde{\psi}^*\nabla\tilde{\psi}). \tag{18}$$

The factors $1/2\mathrm{i}$ have been introduced in order to obtain real expressions for these quantities. The assessed quantity and its flow are equal either for the real linearly independent solutions (3) or for the complex solution

$$\tilde{\psi} = \psi_1 + \psi_2\mathrm{i} \tag{19}$$

together with its concomitant definitions of density and flow (18) as may be seen from direct substitution. The constants b_{1r}, b_{2i} have been set to one since they may always be included within the solutions. This result could be anticipated from (16) since the constants were chosen so that the complementary function is the complex conjugate of the reference function $\tilde{\psi}^{(\mathrm{comp})} = \tilde{\psi}^*$. The expression for the density in terms of complex conjugate fields (18) is encountered when dealing with the Klein–Gordon Schrödinger equation [12]. Nonetheless, as we have mentioned, it is commonly dismissed because it is not positive definite. This fact, together with the lack of relativistic invariance of the positive definite density (44), led Dirac to search for a different wave equation [4]. So far, according to the present results, it is clear that this density corresponds to a quantity assessed between two complementary fields. The relative phase between these fields, as we shall discuss in the following section, defines the sign of this quantity.

In harmonic time-dependent phenomena, evaluation of a product involving the function and its conjugate is often used as a mathematical technique in order to perform an average [13]. The above derivation with a complex field may be misleading because it may be thought that the complex conjugate expressions involve some sort of averaging. However, in the complementary functions procedure with real linearly independent solutions no average was performed at all. Since the evaluation of the density and flow (18) with the complex function (19) is entirely equivalent to the complementary real functions method, it does not imply any sort of averaging.

4. Complementary field evaluation

In the case of a disturbance with arbitrary spatial and time dependence, the independent solution cannot be readily obtained as in second-order differential equations in one variable.

In this general case, given one solution, there are different possibilities that may be pursued in order to obtain a linearly independent solution. Two such possibilities are explored hereafter. In the first subsection, a generalization of the amplitude and phase representation is invoked in order to obtain a linearly independent solution. This proposal generates a flow that is always orthogonal, in the vector sense, to the wave front even in the time-dependent case. In the second subsection, the time derivative of the wave equation is used to produce the second solution. Under certain circumstances the resulting density becomes positive definite. An asset of this approach is that the one dimensional restriction yields a density that is equal to the energy of the discrete system.

4.1. Orthogonal trajectories

The general solution of the wave equation in terms of real amplitude and phase variables is

$$\psi_1(\mathbf{r}, t) = a(\mathbf{r}, t)\cos(s(\mathbf{r}, t) + \varphi_0) \tag{20}$$

where φ_0 is a constant phase. Following a generalization of the previous results, i.e. (10) and (13), the complementary field is proposed to be

$$\psi_2^{(\perp)}(\mathbf{r}, t) = \mp b_\perp a(\mathbf{r}, t)\sin(s(\mathbf{r}, t) + \varphi_0). \tag{21}$$

The complementary field is then a function that is 90° out of phase with respect to the original field. These fields may have arbitrary multiplicative constants; here, the coefficient of $\psi_2^{(\perp)}$ has been set to $b_\perp$ whereas the coefficient of ψ_1 has been normalized to one without loss of generality. The complex solution (19) in terms of amplitude and phase variables from (20) and (21) is

$$\tilde{\psi}^{(\perp)} = a(\mathbf{r}, t)\cos(s(\mathbf{r}, t)) \mp b_\perp a(\mathbf{r}, t)\sin(s(\mathbf{r}, t))\mathrm{i}$$

and its polar representation is

$$\tilde{\psi}^{(\perp)} = a(\mathbf{r}, t)\left[1 + \left(b_\perp^2 - 1\right)\sin^2(s(\mathbf{r}, t))\right]^{\frac{1}{2}}\exp\{\mp\mathrm{i}\arctan[b_\perp\tan(s(\mathbf{r}, t))]\} \tag{22}$$

where the constant phase has been set to zero. If the complementary field constant coefficient is normalized, $b_\perp = 1$, the polar expression for the complex field is simply

$$\tilde{\psi}^{(\perp)} = a(\mathbf{r}, t)\,\mathrm{e}^{\mp\mathrm{i}(s(\mathbf{r}, t)+\varphi_0)}. \tag{23}$$

The terms involving temporal derivatives of the conserved quantity (3) are

$$\psi_1\frac{\partial\psi_2^{(\perp)}}{\partial t} = \left[\mp b_\perp a^2\frac{\partial s}{\partial t}\cos^2(s) \mp b_\perp a\frac{\partial a}{\partial t}\sin(s)\cos(s)\right]$$

and

$$\psi_2^{(\perp)}\frac{\partial\psi_1}{\partial t} = \left[\pm b_\perp a^2\frac{\partial s}{\partial t}\sin^2(s) \mp b_\perp a\frac{\partial a}{\partial t}\sin(s)\cos(s)\right]$$

so that their difference yields the density

$$\psi_\rho^{(\perp)} = \mp b_\perp a^2(\mathbf{r}, t)\frac{\partial s(\mathbf{r}, t)}{\partial t}. \tag{24}$$

If there is no complementary field, i.e. $b_\perp = 0$, the density is obviously zero. It is therefore crucial to have a finite complementary field in order to obtain a non-trivial continuity equation. On the other hand, whether this quantity is positive or negative depends on whether the complementary field leads or lags (by 90°) the reference field (or for that matter, which field is taken as the reference). This issue has been discussed at length in a previous communication in the one-dimensional case [14]. The main drawback of a density which is not positive definite, is that it may be inadequate to represent some variables such as the probability density in certain quantum mechanical problems or the energy density in classical waves. However, the present interpretation shows that the sign of the density relies on whether the complementary field leads or lags the reference field and as such, it may well be an appropriate variable for physical quantities where the equilibrium reference value can be arbitrarily set.

The terms with spatial derivatives in the complementary fields flow (3) follow an analogous derivation

$$\psi_2^{(\perp)}\nabla\psi_1 = \mp b_\perp[-a^2\sin^2(s)\nabla s + a\nabla a\sin(s)\cos(s)]$$

and

$$\psi_1\nabla\psi_2^{(\perp)} = \mp b_\perp[a^2\cos^2(s)\nabla s + a\nabla a\sin(s)\cos(s)]$$

so that

$$\mathbf{S}_\perp = \triangleright\psi_\rho^{(\perp)} = \pm b_\perp a^2(\mathbf{r},t)\nabla s(\mathbf{r},t). \tag{25}$$

The spatially constant phase surfaces define the wave front. Since the flow defined above is zero for a spatially constant phase, the flow $\mathbf{S}_\perp$ is then perpendicular, or orthogonal in the vector sense, to the wave front even in the time-dependent case. To wit, orthogonal trajectories are ensured if $\mathbf{S}\cdot(\nabla\times\mathbf{S}) = 0$ [15]. Evaluating this expression for the complementary fields flow yields

$$\mathbf{S}_\perp\cdot(\nabla\times\mathbf{S}_\perp) = b_\perp^2(a^2\nabla s)\cdot(\nabla\times(a^2\nabla s))$$

but $\nabla\times(a^2\nabla s) = \nabla a^2\times\nabla s$. Thus

$$\mathbf{S}_\perp\cdot(\nabla\times\mathbf{S}_\perp) = a^2\nabla s\cdot(\nabla a^2\times\nabla s) = 0 \tag{26}$$

since the vector $\nabla a^2\times\nabla s$ is orthogonal to ∇s. For a monochromatic wave, the density (24) and flow (25) are time-independent although no averaging process has taken place in the derivation. For a plane wave, the phase spatial dependence is $\phi(\mathbf{r}) = \mathbf{k}\cdot\mathbf{r}$, where the wave vector $\mathbf{k}$ is constant. The assessed quantities are then also spatially constant,

$$\psi_\rho^{(\perp)} = b_\perp a_0^2\omega_0 \qquad \mathbf{S}_\perp = b_\perp a_0^2\mathbf{k} \tag{27}$$

where the upper sign of the expressions has been taken. Therefore, plane wave propagation yields a constant density and flow in this formalism even without performing any averaging. We shall return to this point in the following subsection.

Regarding the parity of these quantities, the density $\psi_\rho^{(\perp)}$ is an odd function of time provided that the constant $b_\perp$ remains invariant under time reversal. This result is not surprising since the field that lags by 90° becomes a leading field by 90° under the time transformation. Under space inversion, $\psi_\rho^{(\perp)}$ remains unaltered. On the other hand, the flow $\mathbf{S}_\perp$ is invariant under time reversal and odd under space inversion.

Introducing the complex disturbance (23) in the scalar wave equation yields

$$\nabla^2 a - (\nabla s\cdot\nabla s)a - \frac{1}{v^2}\left[\frac{\partial^2 a}{\partial t^2} - \left(\frac{\partial s}{\partial t}\right)^2 a\right] = 0 \tag{28}$$

and

$$a\nabla^2 s + 2(\nabla s\cdot\nabla)a - \frac{1}{v^2}\left[a\frac{\partial^2 s}{\partial t^2} + 2\frac{\partial s}{\partial t}\frac{\partial a}{\partial t}\right] = 0. \tag{29}$$

An amplitude a and frequency $\omega\equiv\dot{s}$ time-independent version of these results is often used in the optical scalar theory [16]. The latter equation, provided that the amplitude is finite, may be written as

$$\nabla\cdot(a^2\nabla s) - \frac{1}{v^2}\left[\frac{\partial}{\partial t}\left(a^2\frac{\partial s}{\partial t}\right)\right] = 0. \tag{30}$$

However, this equation is precisely the conservation equation previously derived (4) together with (24) and (25). Therefore, the continuity equation arising from the complementary orthogonal fields is also obtained when a complex disturbance of the form (23) is introduced in the wave equation. For monochromatic electromagnetic fields with linear polarization, this procedure has been shown to yield equivalent results to those obtained from Poynting's theorem [17].

4.2. Derivative field

A second possibility for obtaining an independent solution is the following: in a dispersionless medium, the time derivative of the wave equation is

$$\nabla^2 \frac{\partial \psi}{\partial t} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial \psi}{\partial t} \right) = 0. \tag{31}$$

Since the function $\partial\psi/\partial t$ also satisfies a wave equation, a linearly independent solution may be obtained from the identification $\psi_1 \to \partial\psi/\partial t$ and $\psi_2 \to \psi$. The complementary fields density and flow from (3) are then

$$\psi_\rho^{(\mathrm{DF})} = \left(\frac{\partial \psi}{\partial t} \right)^2 - \psi \frac{\partial^2 \psi}{\partial t^2} \qquad \triangleright\, \psi_\rho^{(\mathrm{DF})} = \psi \nabla \left(\frac{\partial \psi}{\partial t} \right) - \frac{\partial \psi}{\partial t} \nabla \psi. \tag{32}$$

These expressions may be economically written as

$$\psi_\rho^{(\mathrm{DF})} = -\psi^2 \frac{\partial^2 \ln \psi}{\partial t^2} \qquad \triangleright\, \psi_\rho^{(\mathrm{DF})} = \psi^2 \nabla \left(\frac{\partial \ln \psi}{\partial t} \right). \tag{33}$$

From these results it follows that the density $\psi_\rho^{(\mathrm{DF})}$ remains invariant under time reversal and space inversion. On the other hand, the flow $\triangleright\, \psi_\rho^{(\mathrm{DF})}$ is an odd function under time reversal and space inversion. In order to describe the density and flow in amplitude and phase variables, let the solution be written as

$$\psi(\mathbf{r}, t) = \psi_2(\mathbf{r}, t) = a(\mathbf{r}, t) \sin(s(\mathbf{r}, t)). \tag{34}$$

The first term in the density (32) is

$$\left(\frac{\partial \psi}{\partial t} \right)^2 = \left(\frac{\partial a}{\partial t} \right)^2 \sin^2 s + a^2 \left(\frac{\partial s}{\partial t} \right)^2 \cos^2 s + a \frac{\partial a}{\partial t} \frac{\partial s}{\partial t} \sin(2s) \tag{35}$$

whereas the second term is

$$\psi \frac{\partial^2 \psi}{\partial t^2} = \left(a \frac{\partial^2 a}{\partial t^2} - a^2 \left(\frac{\partial s}{\partial t} \right)^2 \right) \sin^2 s + \left(a \frac{\partial a}{\partial t} \frac{\partial s}{\partial t} + \frac{1}{2} a^2 \frac{\partial^2 s}{\partial t^2} \right) \sin(2s). \tag{36}$$

The density is thus

$$\psi_\rho^{(\mathrm{DF})} = a^2 \left[\left(\frac{\partial s}{\partial t} \right)^2 - \frac{\partial^2 \ln a}{\partial t^2} \sin^2 s - \frac{1}{2} \frac{\partial^2 s}{\partial t^2} \sin(2s) \right] \tag{37}$$

and the flow is

$$\triangleright\, \psi_\rho^{(\mathrm{DF})} = a^2 \left[-\nabla s \frac{\partial s}{\partial t} + \nabla \left(\frac{\partial \ln a}{\partial t} \right) \sin^2 s + \frac{1}{2} \nabla \left(\frac{\partial s}{\partial t} \right) \sin(2s) \right]. \tag{38}$$

The complex representations of these quantities are obtained from (18) together with

$$\tilde{\psi} = \frac{\partial \psi}{\partial t} + \psi \mathrm{i}. \tag{39}$$

4.2.1. Positive definite density. The density ψ_ρ is not positive definite as has been mentioned before. However, if the field is harmonic in time (11), the density that arises from a complementary derivative field (32) becomes positive definite since

$$\psi_\rho^{(\mathrm{DF})}(\text{harmonic}) = \left(\frac{\partial\psi}{\partial t}\right)^2 + \omega_0^2\psi^2. \tag{40}$$

Therefore, if the field is decomposed in Fourier components, the contribution of each monochromatic component has a positive definite density in this scheme. In the monochromatic case, the density (and flow) defined from the orthogonal trajectories (24) or the derivative function (37) methods is the same $\psi_\rho^{(\mathrm{DF})} = \psi_\rho^{(\perp)}$ provided that $b_\perp = \omega_0$. In either scheme these quantities are constant in time without having performed any average. The reason lies in the fact that, in both cases, two out-of-phase fields are being invoked. In order to elucidate this point recall, for example, the time-independent one-dimensional harmonic oscillator illustrated by a simple pendulum or a spring. The total time-independent energy arises from two time-dependent out-of-phase functions, namely, the kinetic and potential energy. This result is obtained in the present formalism from the derivative field density (32) by letting ψ represent the displacement of the particle; the complementary field $\psi_1 = \mathrm{d}\psi/\mathrm{d}t$ then stands for the velocity of the particle. The force acting on a mass m executing harmonic motion in one dimension is $F = m\ddot{\psi} = -\kappa\psi$, where κ is the restoring constant. This condition corresponds to the time harmonic field restriction (11) in the continuum case. Thus the invariant density $\psi_\rho^{(\mathrm{DF})} = \dot{\psi}^2 + (\kappa/m)\psi^2$ scaled by a factor of $m/2$ is, in this example, equal to the total energy of the system. In contrast, the energy density usually associated with a scalar wave field (44) does not yield the energy of the discrete harmonic oscillator system in the monochromatic one-dimensional limit.

A second example of a positive definite density is the propagation of a Gaussian pulse. Consider a plane carrier wave with Gaussian temporal envelope

$$\psi(\text{Gauss.}) = a_0 \exp\left[-\frac{\left(\frac{\phi(\mathbf{r})}{\omega_0} - t\right)^2}{b_0^2}\right]\sin[\phi(\mathbf{r}) - \omega_0 t] \tag{41}$$

where b_0 is a constant proportional to the pulse width. The density evaluated from (32) is

$$\psi_\rho^{(\mathrm{DF})}(\text{Gauss.}) = \frac{a_0^2}{b_0^2}\exp\left[-\frac{2\left(\frac{\phi(\mathbf{r})}{\omega_0} - t\right)^2}{b_0^2}\right]\left(2\sin^2[\phi(\mathbf{r}) - \omega_0 t] + b_0^2\omega_0^2\right) \tag{42}$$

which is again a positive definite quantity. Note that an arbitrary transverse spatial amplitude dependence does not alter the positive definite nature of this result since only temporal partial derivatives are involved in the density definition.

5. Comparison with energy density and flow

The continuity equation associated with energy conservation in classical scalar waves or a positive definite probability density in second-order differential equations in quantum mechanics may be derived in a variety of ways. One such procedure is to multiply the dispersionless wave equation by the temporal derivative of the wavefunction [18]. Upon rearrangement of the terms, the conservation equation reads

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{1}{v^2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}\nabla\psi\cdot\nabla\psi\right] + \nabla\cdot\left(-\frac{\partial\psi}{\partial t}\nabla\psi\right) = 0. \tag{43}$$

Thus a density and a flux are defined as

$$\mathcal{E}_d = \underbrace{\frac{1}{2}\frac{1}{v^2}\left(\frac{\partial \psi}{\partial t}\right)^2}_{\text{kinetic}} + \underbrace{\frac{1}{2}\nabla\psi \cdot \nabla\psi}_{\text{potential}} \qquad \mathbf{S}_d = -\frac{\partial \psi}{\partial t}\nabla\psi. \tag{44}$$

These variables are usually translated into quantities with energy and energy flow units through multiplication by the adequate quantity. For example, a mechanical wave where the perturbation corresponds to a displacement would involve a factor ρv^2, where ρ is the mass density. It is customary to associate the two terms in the energy density as the sum of kinetic and potential energy [19].

The purpose in this section is to compare these expressions with their counterparts obtained from the complementary fields approach. The first issue that has already been pointed out is that whereas the density $\mathcal{E}_d$ is positive definite, the densities that arise from the complementary fields are not positive definite but under restricted circumstances. Nonetheless, it should be mentioned that the density $\mathcal{E}_d$ also presents certain drawbacks, for example under Lorentz transformations where it does not lead to a relativistically invariant definition of the integrated probability [20].

The complementary field evaluated with the derivative field yields a density and flow that have the same spatial and temporal parities as the energy density and flow. Namely $\mathcal{E}_d$ and $\psi_\rho^{(\mathrm{DF})}$ are even under time and space inversion whereas $\mathbf{S}_d$ and $\triangleright\ \psi_\rho^{(\mathrm{DF})}$ are odd in either case. For this reason, we shall compare these quantities hereafter. The energy density in terms of the amplitude and phase variables (20) is given by

$$\begin{aligned}\mathcal{E}_d = \frac{1}{2}a^2\left(\frac{1}{v^2}\left(\frac{\partial s}{\partial t}\right)^2 + \nabla s\cdot\nabla s\right)\sin^2 s + \frac{1}{2}\left(\frac{1}{v^2}\left(\frac{\partial a}{\partial t}\right)^2 + \nabla a\cdot\nabla a\right)\cos^2 s \\ - \frac{a}{2}\left(\frac{1}{v^2}\frac{\partial a}{\partial t}\frac{\partial s}{\partial t} + \nabla s\cdot\nabla a\right)\sin(2s).\end{aligned} \tag{45}$$

The corresponding flow is

$$\mathbf{S}_d = \left(-a^2\nabla s\frac{\partial s}{\partial t}\sin^2 s - \nabla a\frac{\partial a}{\partial t}\cos^2 s + \frac{a}{2}\left(\nabla s\frac{\partial a}{\partial t} + \nabla a\frac{\partial s}{\partial t}\right)\sin(2s)\right). \tag{46}$$

The density $\psi_\rho^{(\mathrm{DF})}$ and flow $\triangleright\ \psi_\rho^{(\mathrm{DF})}$ given by equations (37) and (38) respectively have a somewhat similar structure. Nonetheless, the (DF) quantities involve terms that do not exhibit a phase-dependent oscillation as well as second-order derivatives. In the particular case of a plane wave, since the amplitude is constant and the phase is linear in the time and space variables, the energy density and flow are then

$$\mathcal{E}_d = \frac{1}{2}a_0^2\left(\frac{1}{v^2}\omega_0^2 + \mathbf{k}\cdot\mathbf{k}\right)\sin^2(\mathbf{k}\cdot\mathbf{r} - \omega t) \qquad \mathbf{S}_d = a_0^2\omega_0\sin^2(\mathbf{k}\cdot\mathbf{r} - \omega t)\mathbf{k} \tag{47}$$

so that even in this particular condition they are time dependent. In contrast, the complementary field quantities evaluated with the derivative field yield time-independent results

$$\psi_\rho^{(\mathrm{DF})} = a_0^2\omega_0^2 \qquad \triangleright\ \psi_\rho^{(\mathrm{DF})} = a_0^2\omega_0\mathbf{k} \tag{48}$$

just as in the orthogonal trajectories scheme (27). The underlying reason for this state of affairs is that the kinetic and potential energy terms in (44) correspond to the sum of in phase fields whereas the terms in the complementary fields density (32) correspond to the addition of out-of-phase fields.

5.1. Averages

Rather than the instantaneous time-dependent energy density and flow what is frequently measured is the average of these quantities. If the change of the amplitude and the phase temporal derivative are negligible over a period, averages may be performed over the fast varying trigonometric functions. Nonetheless, it should be stressed that this approximation may no longer be valid for ultra-short pulses such as those presently attainable in the femtosecond regime in the optical region [21]. A constant time derivative of the phase within a period is approximated by $s(\mathbf{r}, t) = \phi(\mathbf{r}) - \omega t$, where ω is constant over a period. The average per unit time of the squared trigonometric functions is thus $\omega/2\pi \int_0^{2\pi/\omega} \sin^2(\phi(\mathbf{r}) - \omega t)\, \mathrm{d}t = 1/2$. The average energy density from (45) is then

$$\langle \mathcal{E}_d \rangle = \frac{1}{4} \left(\frac{1}{v^2} a^2 \left(\frac{\partial s}{\partial t} \right)^2 + \frac{1}{v^2} \left(\frac{\partial a}{\partial t} \right)^2 + a^2 \nabla s \cdot \nabla s + \nabla a \cdot \nabla a \right) \tag{49}$$

whereas the average flow is

$$\langle \mathbf{S}_d \rangle = -\frac{1}{2} \left(a^2 \nabla s \frac{\partial s}{\partial t} + \nabla a \frac{\partial a}{\partial t} \right). \tag{50}$$

These expressions are often used in a scalar representation of electromagnetic fields [15]. The first and second pairs of terms in (49) are associated with electric and magnetic energy densities, respectively. Radiometry in the Walther, Marchand and Wolf (WMW) formulation [22] also uses these definitions of density and flow that in terms of a complex scalar $\tilde{\psi} = a \exp(\mathrm{i}s)$ are given by [23]

$$\langle \mathcal{E}_d \rangle = \frac{1}{2} \nabla \tilde{\psi} \cdot \nabla \tilde{\psi}^* + \frac{1}{2} \frac{1}{v^2} \frac{\partial \tilde{\psi}}{\partial t} \frac{\partial \tilde{\psi}^*}{\partial t} \qquad \langle \mathbf{S}_d \rangle = -\frac{1}{2} \left(\frac{\partial \tilde{\psi}}{\partial t} \nabla \tilde{\psi}^* + \frac{\partial \tilde{\psi}^*}{\partial t} \nabla \tilde{\psi} \right). \tag{51}$$

These results should be contrasted with the complementary fields complex representation (18) where no averaging is being performed. On the other hand, the average of the complementary fields density using the derivative field for the linearly independent solution (37) reduces to

$$\left\langle \psi_\rho^{(\mathrm{DF})} \right\rangle = a^2 \left[\left(\frac{\partial s}{\partial t} \right)^2 - \frac{1}{2} \frac{\partial^2 \ln a}{\partial t^2} \right] = a^2 \left(\frac{\partial s}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial a}{\partial t} \right)^2 - \frac{1}{2} a \frac{\partial^2 a}{\partial t^2} \tag{52}$$

and the average flow from (38) is

$$\left\langle \triangleright\, \psi_\rho^{(\mathrm{DF})} \right\rangle = -a^2 \left[\nabla s \frac{\partial s}{\partial t} - \frac{1}{2} \nabla \left(\frac{\partial \ln a}{\partial t} \right) \right] = -a^2 \nabla s \frac{\partial s}{\partial t} - \frac{1}{2} \nabla a \frac{\partial a}{\partial t} + \frac{1}{2} a \nabla \left(\frac{\partial a}{\partial t} \right). \tag{53}$$

The WMW approach employs the spectral flow vector as a starting point for calculating all radiometric quantities. The spectral flow vector is defined as the flow vector (51) restricted to a monochromatic field. Since the monochromatic condition requests a time-independent amplitude, the spectral flow vector in the WMW (50) and the complementary fields (53) formalisms become identical but for a factor of 1/2 that should be included in the latter definition. However, the average energy density (49) is not equal to the average of the complementary fields density (52) in the monochromatic case. Nonetheless, both quantities only differ by a time-independent function that, as we mentioned in section 2, preserves the same conservation equation. If the field is further restricted to a plane wave, then the average density in the two formalisms is the same provided that the complementary fields density is scaled by a factor of $(2v^2)^{-1}$. This factor has not been included in the complementary field definitions in order to allow for the possibility of dealing with appropriate designations for a homogeneous medium with dispersion or an inhomogeneous medium without dispersion (see equation (5) and thereafter). The situation becomes quite different in the presence of finite wave-trains. The amplitude time derivatives are increasingly important as the wave is shortened in time. Furthermore, for very short pulses a coupling of spatial and temporal effects becomes important even for propagation in a non-dispersive medium [25]. Important differences are thus expected in the different schemes in the presence of very short pulses even for the averaged quantities.

6. Conclusions

The conservation equation that arises from the complementary functions procedure has been interpreted in terms of two complementary or out-of-phase fields where the assessed density is interchanged between these two fields. This picture corresponds to the physical idea of a propagating imbalance where there exists an exchange between two forms of energy. The non-definite density may be understood in terms of which of the fields is the leading or the lagging field. Nonetheless, a positive definite density implies that the presence of a wave always leads to an increased density relative to the equilibrium state without waves. However, a negative local density is not necessarily an unphysical result. Let us speculate that if the density represents the energy exchange between the two fields, a negative value could be interpreted in terms of a lower energy density with respect to equilibrium such as a bound state.

The contribution to the density and flow from two linearly independent solutions has been shown to stem from the orthogonal functions; any linearly dependent part has no further contribution to the assessed quantities. The complementary field arising from the linearly independent solution of the wave equation is readily calculated for one degree of freedom. However, in the $(3+1)$-dimensional case, the linearly independent solution cannot be obtained in a straightforward way. Two possibilities have been explored here: (a) a complementary field obtained in an amplitude and phase representation that leads to a wave front orthogonal to the propagating direction, and (b) a complementary field obtained from the derivative of the reference field solution. The table below summarizes the main results in either scheme.

	Density	Flow
General form	$\psi_\rho = \left(\psi_1 \frac{\partial\psi_2}{\partial t} - \psi_2 \frac{\partial\psi_1}{\partial t}\right)$	$\triangleright\ \psi_\rho = (\psi_2 \nabla\psi_1 - \psi_1\nabla\psi_2)$
Complex rep.	$\psi_\rho = \frac{1}{2\mathrm{i}}\left(\tilde{\psi}^* \frac{\partial\tilde{\psi}}{\partial t} - \tilde{\psi}\frac{\partial\tilde{\psi}^*}{\partial t}\right)$	$\triangleright\ \psi_\rho = \frac{1}{2\mathrm{i}}(\tilde{\psi}\nabla\tilde{\psi}^* - \tilde{\psi}^*\nabla\tilde{\psi})$
Orthogonal traj.	$\psi_\rho^{(\perp)} = \mp b_\perp a^2 \frac{\partial s}{\partial t}$	$\triangleright\ \psi_\rho^{(\perp)} = \pm b_\perp a^2 \nabla s$
Derivative field	$\psi_\rho^{(\mathrm{DF})} = -\psi^2 \frac{\partial^2 \ln\psi}{\partial t^2}$	$\triangleright\ \psi_\rho^{(\mathrm{DF})} = \psi^2\nabla\left(\frac{\partial \ln\psi}{\partial t}\right)$
Derivative field in a and s vars.	$a^2\left[\left(\frac{\partial s}{\partial t}\right)^2 - \frac{\partial^2 \ln a}{\partial t^2}\sin^2 s - \frac{1}{2}\frac{\partial^2 s}{\partial t^2}\sin(2s)\right]$	$a^2\left[-\frac{\partial s}{\partial t}\nabla s + \nabla\left(\frac{\partial \ln a}{\partial t}\right)\sin^2 s + \frac{1}{2}\nabla\left(\frac{\partial s}{\partial t}\right)\sin(2s)\right]$

In the general form, the complementary field density and flow are described in terms of two real linearly independent solutions ψ_1, ψ_2. Appropriate definitions in terms of a complex disturbance $\tilde{\psi} = \psi_1 + \mathrm{i}\psi_2$ yield identical results to those obtained with the real solutions. The complex formalism, although it involves complex conjugate fields, does not entail any sort of averaging. The quantities evaluated with the generalized out-of-phase complementary field are adequately represented in amplitude and phase variables. In this orthogonal trajectories scheme the flow is always perpendicular to the constant phase surfaces. In addition, the density reduces to a previously known TDHO exact invariant in the one-dimensional limit. On the other hand, the quantities evaluated with the derivative field formalism may be expressed in terms of the wavefunction or amplitude and phase variables. The density defined in this way is positive definite under restricted circumstances, for example, for a monochromatic wave or a pulse with Gaussian temporal envelope. The assessed quantities evaluated with the time derivative complementary field have been compared with the usual scalar field energy and momentum flow definitions since they both share the same spatial and temporal parities. The density evaluated with these two procedures is abridged in the following table.

Density	Usual scalar field definitions	Complementary field
General form	$\mathcal{E}_d = \frac{1}{2}\frac{1}{v^2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}\nabla\psi\cdot\nabla\psi$	$\psi_\rho^{(\mathrm{DF})} = \left(\frac{\partial\psi}{\partial t}\right)^2 - \psi\frac{\partial^2\psi}{\partial t^2}$
Complex rep.	$\mathcal{E}_d = \frac{1}{2}\frac{1}{v^2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}\nabla\psi\cdot\nabla\psi$	$\psi_\rho = \frac{1}{2\mathrm{i}}\left(\tilde{\psi}^*\frac{\partial\tilde{\psi}}{\partial t} - \tilde{\psi}\frac{\partial\tilde{\psi}^*}{\partial t}\right)$
Average $\langle\mathcal{E}_d\rangle$	$\frac{1}{2}\frac{1}{v^2}\frac{\partial\tilde{\psi}}{\partial t}\frac{\partial\tilde{\psi}^*}{\partial t} + \frac{1}{2}\nabla\tilde{\psi}\cdot\nabla\tilde{\psi}^*$	
Average in a and s vars.	$\frac{1}{4}\left\{\frac{1}{v^2}\left[a^2\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial a}{\partial t}\right)^2\right] + a^2\nabla s\cdot\nabla s + \nabla a\cdot\nabla a\right\}$	$\left\langle\psi_\rho^{(\mathrm{DF})}\right\rangle = a^2\left[\left(\frac{\partial s}{\partial t}\right)^2 - \frac{1}{2}\frac{\partial^2 \ln a}{\partial t^2}\right]$

The complementary fields density involves the sum of two out-of-phase terms whereas the usual definitions imply the sum of two in-phase fields as may be seen from the general form of these two expressions. The average of these quantities in the real amplitude and phase representation exhibit similar terms although the complementary field density and flow involve second-order derivative terms. The complementary fields definitions require a constant factor $(2v^2)^{-1}$ in order to obtain the same results as the usual scalar field quantities for a dispersionless monochromatic plane wave. In the propagationless harmonic limit, the complementary field density reduces to the energy of the time-independent harmonic oscillator. In contrast, the usual definition of the wave energy density does not reduce to an invariant in such a limit (only its average is equal to the oscillator energy). The flow assessed with the two procedures is abbreviated in the table below.

Flow	Usual definition	Complementary field
General form	$\mathbf{S}_d = -\frac{\partial\psi}{\partial t}\nabla\psi$	$\triangleright\,\psi_\rho^{(\mathrm{DF})} = \psi\nabla\left(\frac{\partial\psi}{\partial t}\right) - \frac{\partial\psi}{\partial t}\nabla\psi$
Complex rep.	$\mathbf{S}_d = -\frac{\partial\psi}{\partial t}\nabla\psi$	$\triangleright\,\psi_\rho = \frac{1}{2\mathrm{i}}(\tilde{\psi}\nabla\tilde{\psi}^* - \tilde{\psi}^*\nabla\tilde{\psi})$
Average $\langle\mathbf{S}_d\rangle$	$-\frac{1}{2}\left(\frac{\partial\tilde{\psi}}{\partial t}\nabla\tilde{\psi}^* + \frac{\partial\tilde{\psi}^*}{\partial t}\nabla\tilde{\psi}\right)$	
Average in a and s vars.	$-\frac{1}{2}\left(a^2\nabla s\frac{\partial s}{\partial t} + \nabla a\frac{\partial a}{\partial t}\right)$	$\left\langle\triangleright\,\psi_\rho^{(\mathrm{DF})}\right\rangle = \left[-a^2\nabla s\frac{\partial s}{\partial t} - \frac{1}{2}\nabla a\frac{\partial a}{\partial t} + \frac{1}{2}a\nabla\left(\frac{\partial a}{\partial t}\right)\right]$

In the monochromatic case, the flow in either scheme is the same provided that the complementary field flow is scaled by a factor of $1/2$. Whether these two alternative definitions of flow lead to different experimental observations is an issue that requires further discussion. From the previous analysis it is likely that precise measurements with time-dependent pulses should exhibit appreciable differences.

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